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LIMIT BEHAVIOR OF SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS

BY

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ABSTRACT. Consider a system of m stochastic differential equations $d\xi = a(t, \xi)dt + \sigma(t, \xi)dw$. The limit behavior of $\xi(t)$, as $t \to \infty$, is studied. Estimates of the form $E|\xi(t) - \bar{\sigma}w(t)|^2 = O(t^{1-\delta})$ are derived, and various applications are given.

Introduction. Consider a system of m stochastic differential equations

$$d\xi(t) = a(t, \xi(t))\,dt + \sigma(t, \xi(t))\,dw(t) \tag{0.1}$$

where $a = (a_1, \cdots, a_m)$, σ is an $m \times m$ matrix (σ_{ij}), $w(t) = (w_1(t), \cdots, w_m(t))$, and the $w_i(t)$ are independent Brownian motions. The fundamental theory for such a system is presented in detail in [5], [6] (see also [2], [12]). In this paper we are concerned with the behavior of solutions $\xi(t)$ as $t \to \infty$. Our purpose is to find general conditions on a and σ under which the solutions will have asymptotic behavior similar to that of a Brownian motion. Some results in this direction were obtained by Kulinič ([7]–[10]) and in Gihman-Skorohod [6]. These results, however, are derived only for the case of one stochastic equation with, usually, time independent coefficients a, σ.

In §1 we prove, under very weak assumptions, that every solution of (0.1) is unbounded with probability 1. The main results of this paper are given in §3. They are concerned with asymptotic estimates on $\xi(t)$. The most interesting estimates (given in Theorem 5) are of the form

$$E|\xi(t) - \bar{\sigma}w(t)|^2 = o(t^{1-\delta}) \qquad (0 \le \delta < 1), \tag{0.2}$$

$$E|\xi(t) - \bar{\sigma}w(t)|^2 = O(1). \tag{0.3}$$

The estimate (0.2) is proved under the following assumptions: The diffusion matrix $b(t, x) = \frac{1}{2}\sigma(t, x)\sigma^*(t, x)$ is positive definite, uniformly for $t \ge 0$, x in compact sets,

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$$|a_i(t, x)| \leq \epsilon(|x|)/(1 + |x|)^{1+\delta}, \tag{0.4}$$

$$|\sigma_{ij}(t, x) - \bar{\sigma}_{ij}| \leq \epsilon(|x|)/(1 + |x|)^{\delta} \tag{0.5}$$

where $\epsilon(r) \to 0$ if $r \to \infty$ and $\bar{\sigma}_{ij}$ are constants, and the matrix $\bar{b} = \frac{1}{2}\bar{\sigma}\bar{\sigma}^*$ has at least two positive eigenvalues. If (0.4), (0.5) hold with $\delta > 1$ and if $\bar{b}$ has at least three positive eigenvalues then (0.3) holds.

Some auxiliary estimates needed in §3 are proved in §2.

In §4 we give some applications of the estimates (0.2), (0.3).

In §5 we consider the case $m = 1$. It is shown that if $\sigma \equiv 1$ and $a = a(x)$ is in $L^1(-\infty, \infty)$, then the estimate (0.2) with $\delta = 0$ is true if and only if

$$\int_{-\infty}^{\infty} a(x)\,dx = 0.$$

The case of a general $\sigma = \sigma(x)$ is similarly treated.

In §6 we give an example of a system, with σ = identity matrix, for which $a(x) = O((\log|x|)/|x|)$, but for which the estimate

$$E|\xi(t)|^2 = O(t)$$

is false.

In §7 we consider equations with unbounded a, σ and briefly extend the estimates of §§2, 3.

Acknowledgement. In the original version of the manuscript, the proofs of Theorems 2–5 involved comparison functions constructed by actually solving boundary value problems. Professor S. R. S. Varadhan has pointed out that much simpler comparison functions can be used and that this leads to both much shorter proofs as well as far less restrictive assumptions on the coefficients. The present version of Theorems 2–5 and their proofs (§§2, 3) are based on the suggestions of Professor Varadhan.

1. **Unboundedness of the solution.** We shall deal with the behavior, as $t \to \infty$, of solutions $\xi(t)$ of stochastic differential systems of m equations

$$d\xi(t) = a(t, \xi(t))\,dt + \sigma(t, \xi(t))\,dw(t) \tag{1.1}$$

with

$$\xi(0) = \xi_0; \tag{1.2}$$

here $a = (a_1, \cdots, a_m)$, $\sigma = (\sigma_{ij})$ is an $m \times m$ matrix and $w(t) = (w_1(t), \cdots, w_m(t))$ where the $w_i(t)$ are independent Brownian motions. The sample space will be denoted by $(\Omega, \mathcal{F}, P)$, and a point in Ω will be denote by ω.

We introduce the diffusion matrix

$$b(t, x) = \tfrac{1}{2}\sigma(t, x)\sigma^*(t, x) \qquad (\sigma^* = \text{transpose of } \sigma) \tag{1.3}$$

and set $b(t, x) = (b_{ij}(t, x))$.

We shall use the notation

$$|C| = \left\{\sum_{i,j}(c_{ij})^2\right\}^{1/2}$$

where C is the matrix (c_{ij}).

Suppose the following condition holds:

(A_0) $a(t, x)$, $\sigma(t, x)$ are measurable functions and, for any $T > 0$, $R > 0$, there are positive constants H_T, H_{TR} such that

$$|a(t, x)| + |\sigma(t, x)| \le H_T(1 + |x|) \quad \text{if } t \ge 0,\ x \in R^m,$$

$$|a(t, x) - a(t, \bar{x})| + |\sigma(t, x) - \sigma(t, \bar{x})| \le H_{TR}|x - \bar{x}|$$
$$\text{if } t \ge 0,\ |x| \le R,\ |\bar{x}| \le R.$$

Then (see [5], [6]) for any ξ_0 with $E|\xi_0|^2 < \infty$ which is independent of $w(t)$, $t \ge 0$, there exists a unique solution of (1.1), (1.2). Another existence theorem [not requiring the Lipschitz continuity of a, σ but assuming a, σ to be bounded, σ continuous and $\sigma\sigma^*$ positive definite] is given in [12].

Throughout this paper it is always assumed that $a(t, x)$, $\sigma(t, x)$ are measurable and $\xi(t)$ is a solution of (1.1), (1.2) with $E|\xi(t)|^2 < \infty$ for all $t \ge 0$. Except for this section, we shall not make the assumption (A_0).

We shall need the following conditions:

(A_1) $a(t, x)$ and $\sigma(t, x)$ are Hölder continuous in (t, x), uniformly in compact subsets of $[0, \infty) \times R^m$.

(A_2) For any $R > 0$ there are positive constants γ_R, Γ_R such that, for some i, $1 \le i \le m$,

$$b_{ii}(t, x)\Gamma_R + a_i(t, x) \ge \gamma_R \quad \text{for all } t \ge 0,\ |x| \le R.$$

The last inequality is satisfied, for instance, if $b_{ii}(t, x) \ge \lambda$, $a_i(t, x) \le C$, where λ, C are positive constants.

Theorem 1. *Assume that* (A_0)–(A_2) *hold and that* $b(t, x)$ *is a positive definite martix for any* $(t, x) \in [0, \infty) \times R^m$. *Let* $\xi(t)$ *be the solution of* (1.1), (1.2). *Then*

$$\limsup_{t\to\infty} |\xi(t)| = \infty \quad a.s. \tag{1.4}$$

Note that since $\xi(t)$ is a continuous process, the assertion (1.4) is equivalent to the assertion that

$$\sup_{t>0} |\xi(t)| = \infty \quad \text{a.s.} \tag{1.5}$$

Proof. Denote by μ the probability measure in R^m of the random variable ξ_0. We first consider the case that μ has compact support K. Denote by B any closed ball in R^m containing K. Denote by ∂B the boundary of B. Denote by $\tau(B)$ the exit time from B of the solution $\xi(t)$, that is,

$\tau(B)$ = first $\bar{t}$ such that $\xi(t) \in B$ if $t \leq \bar{t}$, but $\xi(t_n) \notin B$ for a sequence $\{t_n\}$, $t_n \searrow \bar{t}$ if $n \nearrow \infty$;

$\tau(B) = \infty$ if no such $\bar{t}$ exists.

For any $T > 0$, let

$$\tau_T(B) = \min\{\tau(B), T\}. \tag{1.6}$$

Set $B_T = \{t = T\} \times B$, $S_T = \{0 < t < T\} \times \partial B$, and let $\phi(t, x)$ be the solution of

$$L\phi \equiv \frac{\partial \phi}{\partial t} + \sum_{i,j=1}^{m} b_{ij}(t, x) \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \sum_{i=1}^{m} a_i(t, x) \frac{\partial \phi}{\partial x_i} = -1 \quad \text{in } [0, T) \times B,$$

(1.7)

$$\phi = 0 \quad \text{on } B_T \cup S_T;$$

the existence and uniqueness of ϕ is proved in [4].

We shall prove that

$$E\tau_T(B) = \int_K \phi(0, x)\, d\mu(x). \tag{1.8}$$

By Ito's formula, if $\psi(t, x)$ is a function such that ψ_t, $D_x\psi$, $D_x^2\psi$ are continuous for $(t, x) \in [0, T] \times R^m$ $(D_x\psi = \text{grad}_x \psi,\ D_x^2\psi = \text{grad}_x(D_x\psi))$, then, for any $0 < t \leq T$,

$$\psi(t, \xi(t)) - \psi(0, \xi_0) = \int_0^t D_x \psi(s, \xi(s)) \cdot \sigma(s, \xi(s))\, dw(s)$$

$$+ \int_0^t \left[\sum_{i=1}^{m} a_i(s, \xi(s)) \frac{\partial \psi(s, \xi(s))}{\partial x_i} \right.$$

$$\left. + \sum_{i,j=1}^{m} b_{ij}(s, \xi(s)) \frac{\partial^2 \psi(s, \xi(s))}{\partial x_i \partial x_j} + \frac{\partial \psi(s, \xi(s))}{\partial s} \right] ds.$$

Substituting $t = \tau_T(B)$ we get

$$\psi(\tau_T(B), \xi(\tau_T(B))) - \psi(0, \xi_0)$$

(1.10)

$$= \int_0^{\tau_T(B)} D_x\psi(s, \xi(s)) \cdot \sigma(s, \xi(s))\, dw(s) + \int_0^{\tau_T(B)} (L\psi)(s, \xi(s))\, ds.$$

This formula remains true if ψ_t, $D_x\psi$, $D_x^2\psi$ are only assumed to be continuous in $[0, T] \times B_0$ where B_0 is a ball such that int $B_0 \supset B$. Indeed, denote by $\tilde{\psi}$ a function that agrees with ψ in $[0, T] \times B$ and for which $\tilde{\psi}_t$, $D_x\tilde{\psi}$, $D_x^2\tilde{\psi}$ are continuous in $[0, T] \times R^m$. Since (1.10) holds with ψ replaced by $\tilde{\psi}$, and since each term in (1.10) is equal to the corresponding term with ψ replaced by $\tilde{\psi}$, (1.10) follows for ψ.

We shall now use (1.10) with $\psi = \psi_\epsilon$ where ψ_ϵ is the solution of (1.7) with

B replaced by the ball B_ϵ which is concentric with B and has radius $r_0+\epsilon$, $r_0=$ radius of B. Taking the expectation on both sides of (1.10), we then get

$$(1.11)\quad E\psi_\epsilon(\tau_T(B), \xi(\tau_T(B))) - E\psi_\epsilon(0, \xi_0) = E\int_0^{\tau_T(B)} (-1)\,ds = -E\tau_T(B);$$

note that the expectation of the stochastic integral vanishes.

By standard arguments from [4] one can show that, as $\epsilon \to 0$, $\psi_\epsilon(t, x) \to \phi(t, x)$ uniformly in $[0, T] \times B$. Hence, (1.11) yields

$$E\phi(\tau_T(B), \xi(\tau_T(B))) - E\phi(0, \xi_0) = -E\tau_T(B).$$

Since $(\tau_T(B), \xi(\tau_T(B)))$ belongs to $B_T \cup S_T$ for all $\omega \in \Omega$, $\phi(\tau_T(B), \xi(\tau_T(B))) = 0$. We thus get $E\tau_T(B) = E\phi(0, \xi_0)$; this is the assertion (1.8).

Now let $\chi(x)$ be a function satisfying, for all $t \geq 0$, $L\chi \leq -1$ in $[0, T] \times B$, $0 \leq \chi(x) \leq C$ in B, where C is a positive constant. If $1 + x_i \leq x_i^0$ for all $(x_1, x_2, \cdots, x_m) \in B$ (i as in (A_2)), then we can take

$$\chi = A(\exp(\alpha x_i^0) - \exp(\alpha x_i))$$

where α, A are suitable positve constants; here we use the condition (A_2).

We have $L(\phi - \chi) \geq 0$ in $[0, T) \times B$, $\phi - \chi \leq 0$ on $B_T \cup S_T$. Hence, by the maximum principle, $\phi - \chi \leq 0$ in $[0, T] \times B$. Consequently,

$$(1.12)\qquad 0 \leq \phi(t, x) \leq C; \qquad C \text{ independent of } T.$$

From (1.8), (1.12) we deduce that $E\tau_T(B) \leq C$.

Now, $\tau_T(B) \nearrow$ if $T \nearrow$. Using the monotone convergence theorem, we conclude that $\tau(B) = \lim_{T\to\infty} \tau_T(B)$ satisfies $E\tau(B) \leq C$. In particular,

$$(1.13)\qquad \tau(B) < \infty \quad \text{a.s.}$$

Now take an increasing sequence of closed balls B_n with center 0 and radii $R_n \to \infty$. Let

$$M_n = \left\{\sup_{t>0} |\xi(t)| > R_n\right\}, \qquad M = \bigcap_{n=1}^{\infty} M_n.$$

Denote by χ_n the indicator function of B_n, and let $\xi_n(t)$ be the solution of (1.1) with the initial condition $\xi_n(0) = \chi_n \xi_0$; here the condition (A_0) is first used. Denote by $\tau_n(B_n)$ the exit time from B_n of the solution $\xi_n(t)$. As is well known (see [6]), $\tau_n(B_n) = \tau(B_n)$ a.s. and $\xi_n(t) = \xi(t)$ if $0 \leq t \leq \tau(B_n)$. Hence,

$$P\left\{\sup_{t>0} |\xi(t)| > R_n\right\} = P\left\{\sup_{t>0} |\xi_n(t)| > R_n\right\},$$

and the right-hand side is equal to 1, by (1.13) with $K = B = B_n$. We conclude that $P(M_n) = 1$. Since the sequence $\{M_n\}$ is monotone decreasing to M,

$$P(M) = \lim_{n\to\infty} P(M_n) = 1.$$

Now, if $\omega \in M$, then $\omega \in M_n$ for all n, so that

$$\sup_{t>0} |\xi(t)| > R_n \text{ for all } n,$$

i.e. (1.5) holds. This completes the proof.

2. **Auxiliary results.**

Theorem 2. *Assume that*

$$\sum_{i=1}^{m} b_{ii}(t, x) \leq C, \qquad \sum_{i=1}^{m} x_i a_i(t, x) \leq C,$$

where C is a positive constant. Then

$$E|\xi(t)|^2 \leq Kt + K' \quad \text{for all } t \geq 0, \tag{2.1}$$

where K, K' are positive constants.

Proof. Using Ito's formula with $\phi(x) = |x|^2$, $x = \xi(t)$ and taking the expectation, we get

$$E|\xi(t)|^2 = E|\xi_0|^2 + E\int_0^t g(s, \xi(s))\, ds$$

where

$$g(t, x) = L\phi = 2\sum b_{ii}(t, x) + 2\sum x_i a_i(t, x).$$

Since, by our assumptions, $g(t, x) \leq C_1$ where C_1 is a positive constant, (2.1) follows.

Notation. We shall denote the eigenvalues of $b(t, x)$ by $\lambda_i(t, x)$, where

$$\lambda_1(t, x) \leq \lambda_2(t, x) \leq \cdots \leq \lambda_m(t, x).$$

Lemma 1. *Assume that*

$$\sum_{i,j} |b_{ij}(t, x)| \leq C \quad \text{for } t \geq 0,\ x \in R^m \ (C \text{ constant}); \tag{2.2}$$

for any $R > 0$ there is $\mu(R) > 0$ such that

$$\sum_{i,j} b_{ij}(t, x)\xi_i\xi_j \geq \mu(R)|\xi|^2 \quad \text{if } t \geq 0,\ |x| \leq R,\ \xi \in R^m; \tag{2.3}$$

$$(1 + |x|)\sum_i |a_i(t, x)| \leq \epsilon(|x|) \text{ for } t \geq 0,\ x \in R^m, \text{ where } \epsilon(r) \to 0 \text{ if } r \to \infty; \tag{2.4}$$

$$\lambda_m(t, x) \geq \gamma > 0 \text{ for } t \geq 0,\ x \in R^m \ (\gamma \text{ constant}). \tag{2.5}$$

Let $\phi(t, x)$ be a bounded measurable function such that $\phi(t, x) = 0$ if $x \notin \Omega$ where Ω is a compact set. Then, for any $\eta > 0$,

$$E\left|\int_0^t \phi(s, \xi(s))\, ds\right| \leq K_1 + K_2 t^{(1+\eta)/2} \tag{2.6}$$

where K_1, K_2 are positive constants. If further, (2.7) $\lambda_{m-1}(t, x) \geq \gamma' > 0$ *for* $t \geq 0$, $x \in R^m$ (γ' *constant*)

Then (2.6) *holds for some* $-1 < \eta < 0$.

Proof. Since a translation in the x-space does not change the assumptions and assertions of the lemma, we may assume that $x = 0$ is not in Ω. But then there is a continuous function $\psi(r)$ such that

$$|\phi(t, x)| \leq \psi(|x|), \tag{2.8}$$

and $\psi(r) = 0$ if $r \leq r_0$ or if $r \geq r_1$ where $0 < r_0 < r_1 < \infty$. We shall construct a function $F(r)$ such that $f(x) = F(|x|)$ is in $C^2(R^m)$ and

$$(Lf)(t, x) \geq \psi(|x|). \tag{2.9}$$

Note first that

$$Lf(r) = A(t, x)F''(r) + (F'(r)/r)[B(t, x) - A(t, x) + C(t, x)] \tag{2.10}$$

where

$$A(t, x) = \frac{1}{r^2}\sum_{i,j} b_{ij}(t, x)x_i x_j,$$

$$B(t, x) = \sum_i b_{ii}(t, x),$$

$$C(t, x) = \sum_i x_i a_i(t, x).$$

Let $\theta(r)$ be a continuous function, vanishing for $r < r_0/2$ and satisfying, for $r \geq r_0$,

$$(1 + \theta(r))A(t, x) \leq B(t, x) + C(t, x). \tag{2.11}$$

Since, for any $R > 0$, $A(t, x) \geq \mu(R) > 0$ if $t \geq 0$, $|x| \leq R$, and since $B(t, x) \geq 0$, $|C(t, x)|$ is bounded, one can certainly construct such a function $\theta(r)$. Noting that

$$\lambda_1(t, x) \leq A(t, x) \leq \lambda_m(t, x), \qquad B(t, x) = \lambda_1(t, x) + \cdots + \lambda_m(t, x)$$

and using (2.5) and (2.4), we can construct $\theta(r)$ such that

(2.12) $\theta(r) = -\eta$ if r is sufficiently large (for any $\eta > 0$).

If (2.7) holds then we can take

(2.13) $\theta(r) = -\eta$ if r is sufficiently large (for some $-1 < \eta < 0$);

in fact, we can take any η such that

$$-\eta < \gamma'/\delta', \qquad \delta' = \limsup_{|x| \to \infty} \left\{\sup_{t \geq 0} \lambda_m(t, x)\right\}.$$

Introduce the functions $I(s) = \int_{r_0}^{s} (\theta(u)/u)du$,

(2.14) $$f(x) = F(r) = \frac{1}{\lambda} \int_0^r e^{-I(s)} ds \int_0^s e^{I(\tau)} \psi(\tau)\, d\tau$$

for some $\lambda > 0$, $r = |x|$. Since F is in C^2 and $F(r) = 0$ if $r \leq r_0$, $f(x)$ is in $C^2(R^m)$. Notice that

(2.15) $$F''(r) + (\theta(r)/r) F'(r) = (1/\lambda)\psi(r)$$

and that $F'(r) \geq 0$. Using (2.10), (2.11) we then get

$$Lf(x) \geq [F''(r) + (\theta(r)/r) F'(r)]\ A(t, x) \geq (A(t, x)/\lambda)\psi(r) \geq \psi(r)$$

if $\lambda = \mu(r_1)$ (so that $A(t, x) \geq \lambda$ if $t \geq 0$, $|x| \leq r_1$).

Having proved (2.9), we now use (2.8) and Ito's formula to get

$$E\left| \int_0^t \phi(s, \xi(s))\, ds \right| \leq E \int_0^t \psi(\xi(s))\, ds$$

$$\leq E\int_0^t (Lf)(s, \xi(s))\, ds = EF(|\xi(t)|) - EF(|\xi_0|) \leq EF(|\xi(t)|).$$

Noting that $F(r) \leq Cr^{1+\eta}$, we obtain

$$E\left| \int_0^t \phi(s, \xi(s)) ds \right| \leq CE|\xi(t)|^{1+\eta} + C \leq C\{E|\xi(t)|^2\}^{(1+\eta)/2} + C$$

for suitable constants C. Using Theorem 2, the assertion (2.6) follows.

Remark 1. If (2.2), (2.3), (2.5) hold and if (2.4) is replaced by

(2.16) $$(1 + |x|)^{1+\delta} \sum_i |a_i(t, x)| \leq \epsilon(|x|), \qquad \delta > 0,\ \epsilon(r) \to + \text{ if } r \to 0,$$

then we can construct $\theta(r)$ such that, for large r, $\theta(r) = -D/r^{\beta}$ for any $0 < \beta \leq \delta$ and a positive constant D. It follows that $F(r) \leq Cr$, so that (2.6) is valid with $\eta = 0$.

Remark 2. If the conditions (2.2), (2.3), (2.16) hold and if (2.5) is replaced by

(2.17) $$\lambda_m(t, x) \geq c/(1 + |x|)^{\delta}, \qquad c > 0,\ \delta > 0,$$

then we can again take $\theta(r)$ such that (2.12) holds. Consequently, (2.6) is valid for any $\eta > 0$. Note that the condition (2.17) is equivalent to the condition

$$\sum_i b_{ii}(t, x) \geq e/(1 + |x|)^{\delta} \quad \text{for some } e > 0.$$

We shall next extend Lemma 1 to the case where $\phi(t, x)$ does not necessarily vanish if $|x|$ is large, but either

(2.18) $$|\phi(t, x) - \Phi| \leq C/(1 + |x|)^{\alpha} \qquad (\alpha > 0)$$

or

(2.19) $$|\phi(t, x) - \Phi| \leq \epsilon(|x|)/(1 + |x|)^{\alpha} \qquad (\alpha \geq 0),\ \epsilon(r) \to 0 \text{ if } r \to 0;$$

here Φ is a constant.

Lemma 2. *Let the conditions* (2.2)–(2.5) *hold and let* $\phi(t, x)$ *be a bounded measurable function satisfying* (2.18). *Then*

$$\left|E \int_0^t \phi(s, \xi(s))ds - \Phi t\right| = O(t^{(1+\eta)/2}) + O(t^{1-\alpha/2}) \tag{2.20}$$

for any $\eta > 0$; *if* (2.7) *also holds then* (2.20) *holds for some* $-1 < \eta < 0$. *If the function* $\phi(t, x)$ *satisfies* (2.19) *instead of* (2.18), *then*

$$E\left|\int_0^t \phi(s, \xi(s))ds - \Phi t\right| = O(t^{(1+\eta)/2}) + o(t^{1-\alpha/2}) \tag{2.21}$$

with the same η *as before.*

Proof. Suppose first that (2.19) holds. Then, if any $\epsilon > 0$, there exists an $R_\epsilon > 0$ such that

$$|\phi(t, x) - \Phi| < \epsilon/r^\alpha \quad \text{if } |x| > R_\epsilon,\ t \geq 0.$$

Let $\chi_\epsilon(x) = 1$ if $|x| < R_\epsilon$, $\chi_\epsilon(x) = 0$ if $|x| \geq R_\epsilon$, and write

$$\begin{aligned} \phi(t, x) - \Phi &= [\phi(t, x) - \Phi]\chi_\epsilon(x) + [\phi(t, x) - \Phi][1 - \chi_\epsilon(x)] \\ &\equiv \phi_1(t, x) + \phi_2(t, x). \end{aligned} \tag{2.22}$$

By Lemma 1,

$$E\left|\int_0^t \phi_1(s, \xi(s))ds\right| = O(t^{(1+\eta)/2}). \tag{2.23}$$

Next, let $\psi(r)$ be a continuous function such that

$$\begin{aligned} \psi(r) &= 0 \quad \text{if } 0 < r < r_0/2, \\ \psi(r) &= \epsilon/r^\alpha \quad \text{if } r > r_0, \end{aligned} \tag{2.24}$$

for some $0 < r_0 < R_\epsilon$. We shall slightly modify the definition of $\theta(r)$ by replacing the condition (2.11) by the stricter condition

$$(1 + \theta(r))A(t, x) \leq \rho B(t, x) + C(t, x) \tag{2.25}$$

where ρ is any positive number < 1. We still have (2.12) for any $\eta = \eta(\rho) > 0$ provided ρ is sufficiently close to 1; if (2.7) holds then we can make (2.13) hold if ρ is sufficiently close to 1.

Let $f(x)$ be defined by (2.14), with $\lambda > 0$ to be determined later. Then, by (2.10), (2.25),

$$\begin{aligned} Lf(x) &\geq \left[F''(r) + \frac{\theta(r)}{r} F'(r)\right]A(t, x) + \frac{F'(r)}{r}(1 - \rho)B(t, x) \\ &\geq \frac{F'(r)}{r}(1 - \rho)B(t, x) \geq \frac{c}{\lambda}\frac{1}{r} e^{-I(r)} \int_{r_0}^r e^{I(s)} \frac{\epsilon}{s^\alpha}\, ds \\ &\geq \frac{c}{\lambda}\frac{\epsilon}{r^{1+\alpha}} e^{-I(r)} \int_{r_0}^r e^{I(s)} ds \end{aligned} \tag{2.26}$$

where c is any positive constant $\leq (1-\rho)B(t, x)$. Since $e^{I(r)} \sim r^{-\eta}$ if $r \longrightarrow \infty$, we get

$$\text{(2.27)} \qquad Lf(x) \geq \frac{c_1}{\lambda} \frac{\epsilon}{r^{\alpha}} \geq |\phi_2(t, x)| \qquad (|x| \geq R_\epsilon)$$

where c_1 is a positive constant (independent of ϵ) provided we take $\lambda \leq c_1$. If $|x| < R_\epsilon$ then clearly $Lf(x) \geq 0 = |\phi_2(t, x)|$.

By (2.27) and Ito's formula,

$$\text{(2.28)} \qquad \begin{aligned} E\left|\int_0^t \phi_2(s, \xi(s))\,ds\right| &\leq E\int_0^t \left|\phi_2(s, \xi(s))\right| ds \\ &\leq E\int_0^t (Lf)(s, \xi(s))\,ds = EF(|\xi(t)|) - EF(|\xi_0|) \leq EF(|\xi(t)|). \end{aligned}$$

Since

$$F(r) \leq \begin{cases} C\epsilon r^{2-\alpha} + C & \text{if } 2-\alpha > 1+\eta, \\ C\epsilon r^{1+\eta} + C & \text{if } 2-\alpha < 1+\eta, \end{cases}$$

and since we may assume that that η is such that $\alpha + \eta \neq 1$, we deduce, after using Hölder's inequality and Theorem 2, that

$$EF(|\xi(t)|) \leq \begin{cases} C\epsilon t^{1-\alpha/2} + C & \text{if } \alpha+\eta < 1, \\ C\epsilon t^{(1+\eta)/2} + C & \text{if } \alpha+\eta > 1. \end{cases}$$

Substituting this into (2.28) and combining with (2.23), (2.22), we get

$$\text{(2.29)} \qquad E\left|\int_0^t \phi(s, \xi(s))\,ds - \Phi t\right| \leq C_0 t^{(1+\eta)/2} + C_1 \epsilon t^{1-\alpha/2};$$

C_1 is a constant independent of ϵ. Since ϵ is arbitrary, (2.29) implies the assertion (2.21) with η replaced by η', for any $\eta' > \eta$. But this clearly completes the proof of (2.21). The proof of (2.20) (for ϕ satisfying (2.18)) is similar.

Remark 1. Let the condition (2.4) be replaced by the stricter condition (2.16). Then we can modify the proof of Lemma 2 with $\epsilon = C$, in case (2.5) holds, taking, for any $0 < \beta \leq \delta$,

$$1 - \rho = (D/2)/r^{\beta}, \qquad \theta(r) = D/r^{\beta},$$

for large r, where D is a positive constant; in (2.14) we replace $\psi(r)$ by $\psi(r)r^{\beta}$, so that in (2.26) α is replaced by $\alpha - \beta$. We conclude that if (2.18) holds then

$$\text{(2.30)} \qquad E\left|\int_0^t \phi(s, \xi(s))\,ds - \Phi t\right| = O(t^{1/2}) + O(t^{1-(\alpha-\beta)/2})$$

for any $\beta > 0$.

Remark 2. Suppose now that (2.5) is replaced by (2.17) and that (2.2), (2.3) and (2.16) hold. Modifying the proof of Lemma 2 (by replacing $\psi(r)$ by $\psi(r)r^{\delta}$ in (2.14) and α by $\alpha - \delta$ in (2.26)) we conclude that if (2.19) holds then

$$E\left|\int_0^t \phi(s, \xi(s))\,ds - \Phi t\right| = O(t^{(1+\eta)/2}) + o(t^{1-(\alpha-\delta)/2}) \tag{2.31}$$

for any $\eta > 0$. If ϕ satisfies (2.18) then (2.31) holds with "o" replaced by "O".

Our next concern is to evaluate more precisely the η occurring in Lemma 2. We shall assume that, for all i, j,

$$\lim_{|x|\to\infty} b_{ij}(t, x) = \bar{b}_{ij} \quad \text{uniformly with respect to } t, \tag{2.32}$$

where $\bar{b}_{ij}$ are constant. We then introduce the notation:

$$d = \text{number of positive eigenvalues of } \bar{b} = (\bar{b}_{ij}). \tag{2.33}$$

Observe that if (2.3) holds then (2.5) is equivalent to $d \geq 1$, and (2.7) is equivalent to $d \geq 2$.

We now note that the assumptions and assertions of Lemma 2 remain unchanged if a nonsingular affine transformation $x \longrightarrow Ax$ is performed. But such a transformation changes $b(t, x)$ into $Ab(t, x)A^*$. We can, therefore, choose A such that

$$\bar{b}_{ij} = 0 \quad \text{if } i \neq j,$$

$$\bar{b}_{ii} = 1 \quad \text{if } i = 1, 2, \cdots, d,$$

$$\bar{b}_{ii} = 0 \quad \text{if } i = d + 1, \cdots, m;$$

$d = 0$ means that $\bar{b}_{ii} = 0$ for all i, and $d = m$ means that $\bar{b}_{ii} = 1$ for all i.

If $d \geq 2$ then we can take, in the proofs of Lemmas 1, 2, $\theta(r) = \nu$ for any $\nu < 1$ and all large r. This leads to the assertions of Lemmas 1, 2 with any $-1 < \eta < 0$.

If $d \geq 3$ then we can take $\theta(r) = 3/2$ for all large r. This leads to the assertion of Lemma 1 with $\eta = -1$. If ϕ satisfies (2.18) then, instead of (2.20), we have

$$E\left|\int_0^t \phi(s, \xi(s))\,ds - \Phi t\right| \leq \begin{cases} O(1) + O(t^{1-\alpha/2}) & \text{if } \alpha \neq 2, \\ O(\log t) & \text{if } \alpha = 2. \end{cases} \tag{2.34}$$

If $d \geq 2$ and if (2.16) holds and

$$\sum_{i,j} |b_{ij}(t, x) - \bar{b}_{ij}| \leq \epsilon(|x|)/(1 + |x|)^{\delta}, \qquad \delta > 0,\ \epsilon(r) \longrightarrow 0 \text{ if } r \longrightarrow \infty, \tag{2.35}$$

then we can take, in the proof of Lemma 1, $\theta(r) = 1 - D/r^{\beta}$ (for any $0 < \beta \leq \delta$) for r large, where D is a positive constant. This leads to the assertion of Lemma 1 with $O(t^{(1+\eta)/2})$ replaced by $O(\log t)$. We can use the same $\theta(r)$ also in the proof of Lemma 2 (cf. Remark 1 following Lemma 2). It follows that if ϕ satisfies (2.18) then

$$E\left|\int_0^t \phi(s, \xi(s))\,ds - \Phi t\right| = O(\log t) + O(t^{1-(\alpha-\beta)/2}) \tag{2.36}$$

for any $\beta > 0$.

If $d \geq 1$ and (2.16), (2.35) hold, then we can take, in the proof of Lemma 1, $\theta(r) = -D/r^{\beta}$ (for any $0 < \beta \leq \delta$) for large r where D is a positive constant. We conclude that Lemma 1 holds with $\eta = 0$. As for Lemma 2, if (2.18) holds, then

$$E\left|\int_0^t \phi(s, \xi(s))\,ds - \Phi t\right| = O(t^{1/2}) + O(t^{1-(\alpha-\beta)/2}) \tag{2.37}$$

for any $\beta > 0$.

We sum up:

Lemma 3. (a) *Let the conditions* (2.2)–(2.4), (2.32) *hold and let* $d \geq 2$. *Let* $\phi(t, x)$ *be a bounded measurable function. If* ϕ *satisfies* (2.18) *then* (2.20) *holds for any* $-1 < \eta < 0$, *and if* ϕ *satisfies* (2.19) *then* (2.21) *holds with any* $-1 < \eta < 0$.

(b) *Let the conditions* (2.2)–(2.4), (2.32) *hold and let* $d \geq 3$. *Let* $\phi(t, x)$ *be a bounded measurable function satisfying* (2.18). *Then* (2.34) *holds.*

(c) *Let the conditions* (2.2)–(2.4), (2.35) *and* (2.16) *hold and let* $d \geq 2$. *Let* $\phi(t, x)$ *be a bounded measurable function satisfying* (2.18). *Then* (2.36) *holds.*

(d) *Let the conditions* (2.2)–(2.4), (2.16), (2.35) *hold and let* $d \geq 1$. *Let* $\phi(t, x)$ *be a bounded measurable function satisfying* (2.18). *Then* (2.37) *holds.*

3. Asymptotic estimates.

Theorem 3. *Let the conditions* (2.2)–(2.5) *hold. Then*

$$E|\xi(t)|^2 \geq Kt - K' \quad \textit{for all } t \geq 0, \tag{3.1}$$

where K, K' *are positive constants.*

Proof.

$$L|x|^2 = 2\sum b_{ii}(t, x) + 2\sum x_i a_i(t, x) \geq \gamma - \phi(x)$$

where $\phi(x)$ is a bounded, measurable and nonnegative function having compact support. Hence,

$$E|\xi(t)|^2 - E|\xi_0|^2 = E\int_0^t (L|x|^2)(s, \xi(s))\,ds \geq \gamma t - E\int_0^t \phi(\xi(s))\,ds$$

and, by Lemma 1,

$$E\int_0^t \phi(\xi(s))\,ds = o(t).$$

This yields the inequality (3.1).

Remark. Lemma 1 and Theorem 1 remain true if the condition (2.4) is replaced by the weaker condition

$$(1 + |x|)\sum_i |a_i(t, x)| \leq \epsilon_0, \tag{3.2}$$

ϵ_0 a sufficiently small, positive constant.

Theorem 4. *Let* (2.2), (2.3), (2.16), (2.32) *hold and let* $d \geq 1$. *Then*

$$E|\xi(t)|^2 = 2(\operatorname{tr} \bar{b})\, t + O(t^{(1+\eta)/2}) + o(t^{1-\delta/2}) \tag{3.3}$$

where $\operatorname{tr} \bar{b} = \Sigma_i \bar{b}_{ii}$ *and* η *is any positive number; if* $d \geq 2$ *then* η *is any number* > -1.

When $\delta = 0$ we get

Corollary. *Let* (2.2)–(2.4), (2.32) *hold and let* $d \geq 1$. *Then*

$$E|\xi(t)|^2 = 2(\operatorname{tr} \bar{b})\, t + o(t). \tag{3.4}$$

Proof of Theorem 4. Clearly,

$$|L|x|^2 - 2 \operatorname{tr} \bar{b}| \leq \epsilon'(|x|)/(1 + |x|)^\delta$$

where $\epsilon'(r) \to 0$ if $r \to \infty$. Hence, by Lemmas 2 and 3(a) with $\phi(t, x) = |x|^2$, $\Phi = 2 \operatorname{tr} \bar{b}$,

$$\begin{aligned} E|\xi(t)|^2 - E|\xi_0|^2 &= E\int_0^t (L|x|^2)(s, \xi(s))\, ds \\ &= 2(\operatorname{tr} \bar{b})\, t + O(t^{(1+\eta)/2}) + O(t^{1-\delta/2}) \end{aligned}$$

where η is any positive number if $d \geq 1$ and any number > -1 if $d \geq 2$. This yields (3.3).

Remark 1. Using Lemma 3 one can give variants of (3.3) in case (2.16) holds with $\epsilon(r)$ replaced by a positive constant.

Remark 2. By a slight change in the proof of Theorem 4 one derives the estimate

$$E|\langle e, \xi(t)\rangle|^2 = 2\langle e, \bar{b}e\rangle + O(t^{(1+\eta)/2}) + o(t^{1-\delta/2}) \tag{3.5}$$

where e is any unit vector.

Lemma 4. *Let the conditions* (2.2), (2.3), (2.32) *and* (2.16) *with* $0 \leq \delta < 1$ *hold, and let* $d \geq 2$. *Then*

$$E\left|\int_0^t a_j(s, \xi(s))\, ds\right|^2 = o(t^{1-\delta}). \tag{3.6}$$

Proof. For any $\epsilon > 0$ we can write

$$|a_j(t, x)| \leq g_1(x) + g_2(|x|)$$

where the g_i are bounded measurable functions; $g_1(x)$ has a compact support,

$$g_2(r) = \epsilon/(1 + r)^{1+\delta} \quad \text{if } r > r_1, \tag{3.7}$$

and $g_2(r) = 0$ if $r \leq r_1$ for some $0 < r_1 < \infty$. We have

$$E\left|\int_0^t a_j(s, \xi(s))\, ds\right|^2 \leq 2E\left|\int_0^t g_1(\xi(s))\, ds\right|^2 + 2E\left|\int_0^t g_2(|\xi(s)|)\, ds\right|^2. \tag{3.8}$$

Let $F(r)$ be the function constructed in the proof of Lemma 1 for $\phi = g_1$. Then, for large r,

$$F(r) \le cr^{1+\eta}, \qquad F'(r) \le Cr^{\eta} \tag{3.9}$$

for any $-1 < \eta < 0$. Hence, by Ito's formula,

$$\left| \int_0^t g_1(\xi(s))\, ds \right| \le \int_0^t (LF)(s, \xi(s))\, ds$$

$$\le F(|\xi(t)|^{1+\eta}) - \int_0^t \nabla_x F \cdot \sigma(s, \xi(s))\, dw(s),$$

so that

$$\begin{aligned} E\left| \int_0^t g_1(\xi(s))\, ds \right|^2 &\le 2EF(|\xi(t)|)^2 + 2E \int_0^t |\nabla_x F \cdot \sigma|^2\, ds \\ &\le Ct^{1+\eta} + C + 2E \int_0^t |\nabla_x F \cdot \sigma|^2\, ds. \end{aligned} \tag{3.10}$$

Now, for large r,

$$|\nabla_x F \cdot \sigma|^2 \le Cr^{2\eta} \qquad (-1 < \eta < 0). \tag{3.11}$$

Hence, by Lemma 3(a) with $\alpha = -2\eta$,

$$E \int_0^t |\nabla_x F \cdot \sigma|^2\, ds \le Ct^{1+\eta} + C, \tag{3.12}$$

and (3.10) gives

$$E\left| \int_0^t g_1(\xi(s))\, ds \right|^2 \le Ct^{1+\eta} + C. \tag{3.13}$$

Next let $F_0(r)$ be the function $F(r)$ constructed in Lemma 2 for $\phi_2 = g_2$. We can take $-1 < \eta < -\delta$. Then

$$F_0(r) \le C\epsilon r^{1-\delta} + C\epsilon, \qquad F_0'(r) \le C\epsilon r^{-\delta} + C\epsilon, \tag{3.14}$$

where C is a constant independent of ϵ. Analogously to (3.10), (3.11) we get

$$E\left| \int_0^t g_2(|\xi(s)|)\, ds \right|^2 \le C\epsilon t^{1-\delta} + C + 2E \int_0^t |\nabla_x F_0 \cdot \sigma|^2\, ds, \tag{3.15}$$

$$|\nabla_x F_0 \cdot \sigma|^2 \le C\epsilon^2 / r^{2\delta}. \tag{3.16}$$

By Lemma 3(a),

$$E \int_0^t |\nabla_x F_0 \cdot \sigma|^2\, ds \le Ct^{(1+\eta)/2} + C\epsilon^2 t^{1-\delta} + C. \tag{3.17}$$

Hence,

$$E\left| \int_0^t g_2(|\xi(s)|)\, ds \right|^2 \le C\epsilon t^{1-\delta} + Ct^{(1+\eta)/2} + C. \tag{3.18}$$

Combining this with (3.13), (3.8), we get

$$E\left| \int_0^t a_j(s, \xi(s))\, ds \right|^2 \le Ct^{1+\eta} + Ct^{1-\delta} + C.$$

This gives the assertion of the lemma.

We shall now modify the proof of Lemma 4 in the case where $d \geq 3$ and (2.16) holds with $\delta > 1$.

The function $F(r)$ occurring in (3.9) is given by

$$F(r) = \int_{\rho_0}^{r} e^{-I(s)} ds \int_{\rho_0}^{s} e^{I(\tau)} \bar{\gamma}(\tau) d\tau \quad \text{if } r > \rho_0,$$

where $\bar{\gamma}(r) = 0$ if $r > \rho_1$, for some $0 < \rho_0 < \rho_1 < \infty$; note that r is not to be taken as $|x|$ but rather as $|x - x^0|$ for some x^0 outside the support of $g_1(x)$. We can take $\eta(r) = 3/2$ for r large, so that $I(r) \sim \log r^{3/2}$. Hence

$$F(r) \leq C, \qquad F'(r) \leq C/r^{3/2} \tag{3.19}$$

instead of (3.9). Hence (3.11) is replaced by

$$|\nabla_x F \cdot \sigma| \leq C/r^3. \tag{3.20}$$

Lemma 3(b) with $\alpha = 3$ gives

$$E\int_0^t |\nabla_x F \cdot \sigma|^2 ds = O(1), \tag{3.21}$$

which replaces (3.12). We conclude that (instead of (3.13)

$$E\left| \int_0^t g_1(\xi(s)) ds \right|^2 \leq C. \tag{3.22}$$

Let $0 < r_0 < r_1$ and let $\bar{g}_2(r)$ be a continuous function satisfying $\bar{g}_2(r) = \epsilon/(1 + r)^{1+\delta}$ if $r > r_0$, $\bar{g}_2(r) = 0$ if $r < r_0/2$. Take

$$F_0(r) = C \int_0^r e^{-I(s)} \int_0^r e^{I(\tau)} \bar{g}_2(\tau) d\tau \qquad (C > 0).$$

Since $\delta > 1$,

$$F_0(r) \leq C, \qquad F_0'(r) \leq C/r^k \quad (k = \min(\delta, 3/2)) \tag{3.23}$$

instead of (3.14). It follows that

$$E\left| \int_0^t g_2(|\xi(s)|) ds \right|^2 \leq C + 2E \int_0^t |\nabla_x F_0 \cdot \sigma|^2 ds \tag{3.24}$$

and by, Lemma 3(b) with $\alpha = 2\delta$, $E\int_0^t |\nabla_x F_0 \cdot \sigma|^2 ds \leq C$. Thus, the left-hand side of (3.24) is bounded by a constant. Combining this with (3.22), (3.8) we obtain

Lemma 5. *Let the conditions* (2.2), (2.3), (2.32) *hold, let* $d \geq 3$ *and let* (2.16) *hold with some* $\delta > 1$. *Then*

$$E\left| \int_0^t a_j(s, \xi(s)) ds \right|^2 = O(1). \tag{3.25}$$

Remark 1. If in Lemma 4 we replace the condition (2.16) by

$$\sum_i |a_i(t, x)| \leq C/(1 + |x|)^{1+\delta} \tag{3.26}$$

where $0 < \delta < 1$, then

$$E\left|\int_0^t a_j(s, \xi(s))\,ds\right|^2 = O(t^{1-\delta}). \tag{3.27}$$

Remark 2. If in Lemma 4 we replace (2.16) by (3.26) with $\delta = 1$ and if either $d \geq 3$ or $d \geq 2$ and

$$\sum_{i,j} |b_{ij}(t, x) - \bar{b}_{ij}| \leq C/(1 + |x|)^{\beta} \quad \text{for some } \beta > 0, \tag{3.28}$$

then, by the proofs of Lemmas 4, 5 and upon applying Lemma 3(c), we find that

$$E\left|\int_0^t a_j(s, \xi(s))\,ds\right|^2 = O(\log t). \tag{3.29}$$

We shall need the following condition:

$$\sum_{i,j} |\sigma_{ij}(t, x) - \bar{\sigma}_{ij}| \leq \epsilon(|x|)/(1 + |x|)^{\delta}, \qquad \delta \geq 0, \ \epsilon(r) \to 0 \text{ if } r \to \infty, \tag{3.30}$$

where $\bar{\sigma}_{ij}$ are constants. Note that (3.30) implies (2.35) with the same δ (but not the same $\epsilon(r)$) and that

$$\bar{b} = \tfrac{1}{2}\bar{\sigma}\bar{\sigma}^*, \qquad 2 \operatorname{tr} \bar{b} = |\bar{\sigma}|^2.$$

We shall also need the condition

$$\sum_{i,j} |\sigma_{ij}(t, x) - \bar{\sigma}_{ij}| \leq C/(1 + |x|)^{\delta}, \qquad \delta > 0. \tag{3.31}$$

This condition implies (3.28) with $\beta = \delta$.

We can now state the main results on asymptotic estimates.

Theorem 5. (a) *Let* (2.2), (2.3), (2.16), (3.30) *hold with* $0 \leq \delta < 1$, *and let* $d \geq 2$. *Then*

$$E|\xi(t) - \bar{\sigma}w(t)|^2 = o(t^{1-\delta}). \tag{3.32}$$

If (2.16), (3.30) *are replaced by* (3.26), (3.31) *and* $0 < \delta < 1$, *then*

$$E|\xi(t) - \bar{\sigma}w(t)|^2 = O(t^{1-\delta}). \tag{3.33}$$

(b) *Let* (2.2), (2.3), (2.16), (3.30) *hold with* $\delta = 1$ *and let* $d \geq 3$. *Then*

$$E|\xi(t) - \bar{\sigma}w(t)|^2 = o(\log t). \tag{3.34}$$

If (2.16), (3.30) *are replaced by* (3.26), (3.31) *and* $\delta = 1$, *then*

$$E|\xi(t) - \bar{\sigma}w(t)|^2 = O(\log t). \tag{3.35}$$

(c) *Let* (2.2), (2.3), (3.26), (3.31) *hold for some* $\delta > 1$ *and let* $d \geq 3$. *Then*

$$E|\xi(t) - \bar{\sigma}w(t)|^2 = O(1). \tag{3.36}$$

Proof. Let the conditions (2.2), (2.3), (2.16), (3.30), $0 \leq \delta < 1$, $d \geq 2$, of (a) hold. Consider

$$E\left|\int_0^t (\sigma-\bar\sigma)\,dw(s)\right|^2 = E\int_0^t |\sigma-\bar\sigma|^2 ds.$$

Since $|\sigma(t, x)-\bar\sigma|^2 \le \epsilon^2(|x|)/(1+|x|)^{2\delta}$, Lemma 3(a) implies that

$$E\int_0^t |\sigma(s, \xi(s))-\bar\sigma|^2 ds = o(t^{1-\delta}).$$

Hence

$$E\left|\int_0^t (\sigma(s, \xi(s))-\bar\sigma)\,dw(s)\right|^2 = o(t^{1-\delta}). \tag{3.37}$$

Writing

$$\xi(t)-\bar\sigma w(t) = \xi_0 + \int_0^t a(s, \xi(s))ds + \int_0^t [\sigma(s, \xi(s))-\bar\sigma]dw(s)$$

and using (3.37) and Lemma 4, the assertion (3.32) follows.

The proofs of (3.33)–(3.36) are similar. In proving (3.33)–(3.35) we use Remarks 1, 2 following Lemma 5, and in the proof of (3.36) we employ Lemma 5.

Remark. In all the results of §§2, 3 we have assumed the condition (2.3). By slight changes in the proofs one can verify that if

$$(1+|x|)\sum_i |a_i(t, x)| \le \epsilon \qquad \text{for all } t \ge 0,\ x \in R^m$$

where ϵ is sufficiently small, then the condition (2.3) may be omitted.

4. **Applications of the asymptotic estimates.** For convenience we restate some of the assumptions made in Theorem 5.

$$\lim_{|x|\to\infty} \sigma_{ij}(t, x) = \bar\sigma_{ij} \qquad \text{uniformly with respect to } t,\ 1 \le i,\ j \le m, \tag{4.1}$$

$$\lim_{|x|\to\infty} (1+|x|)\sum_i |a_i(t, x)| = 0 \qquad \text{uniformly with respect to } t. \tag{4.2}$$

Theorem 5(a) with $\delta = 0$ states

If (2.2), (2.3), (4.1), (4.2) hold and if $d \ge 2$ then

$$E|\xi(t)-\bar\sigma w(t)|^2 = o(t). \tag{4.3}$$

From (4.3) if follows that $\xi(t)/\sqrt{t} - \bar\sigma w(t)/\sqrt{t} \to 0$ in L^2; consequently also in probability. This immediately yields the following theorem on convergence in distribution of $\xi(t)$:

Theorem 6. *Let the assumptions* (2.2), (2.3), (4.1), (4.2) *hold, let* $\bar\sigma$ *be non-singular matrix, and let* $m \ge 2$. *Then*

$$\lim_{t\to\infty} P\{\xi(t) < x\sqrt{t}\}$$
$$= \frac{1}{(2\pi)^{m/2}\det\bar\sigma}\int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_m}\exp\left\{-\frac{1}{2}\sum_{i,j=1}^m \hat b_{ij}y_iy_j\right\}dy_1\cdots dy_m \tag{4.4}$$

where $\hat{b}$ is the inverse matrix to $\bar{b}$.

Suppose next that (2.2), (2.3) hold and that

$$\sum_{i,j} |\sigma_{ij}(t, x) - \bar{\sigma}_{ij}| \le C/(1 + |x|)^{\delta}, \tag{4.5}$$

$$\sum_{i} |a_i(t, x)| \le C/(1 + |x|)^{1+\delta} \tag{4.6}$$

for some $0 < \delta < 1$. Suppose $\bar{\sigma}$ is nonsingular, $m \ge 2$, and denote by $\hat{\sigma}$ the inverse of $\bar{\sigma}$. Then

$$\begin{aligned}\frac{\hat{\sigma}\xi(t) - w(t)}{\sqrt{2t \log \log t}} &= \frac{\hat{\sigma}}{\sqrt{2t \log \log t}} \int_0^t a(s, \xi(s))\, ds \\ &+ \frac{\hat{\sigma}}{\sqrt{2t \log \log t}} \int_0^t [\sigma(s, \xi(s)) - \bar{\sigma}]\, dw(s) + \frac{\hat{\sigma}\xi_0}{\sqrt{2t \log \log t}} \\ &\equiv J_1(t) + J_2(t) + J_3(t) \equiv J(t).\end{aligned} \tag{4.7}$$

We shall denote various positive constants by the same symbol C. By the proof of Theorem 5(a), if $t_n = n^{\lambda}$, $\lambda = 4/\delta$, n large,

$$\begin{aligned}P\left\{\sup_{t_n \le t \le t_{n+1}} |J_1(t)| > \frac{1}{n}\right\} &\le P\left\{\frac{C}{\sqrt{t_n}} \int_0^{t_{n+1}} |a(s, \xi(s))|\, ds > \frac{1}{n}\right\} \\ &\le \frac{Cn^2}{t_n} E\left|\int_0^{t_{n+1}} |a(s, \xi(s))|\, ds\right|^2 \le \frac{Cn^2}{t_n} (t_{n+1})^{1-\delta} \le \frac{C}{n^2}.\end{aligned}$$

Next, by the matringale inequality and the proof of Theorem 5(a),

$$\begin{aligned}&P\left\{\sup_{t_n \le t \le t_{n+1}} |J_2(t)| > \frac{1}{n}\right\} \\ &\qquad \le P\left\{\frac{C}{\sqrt{t_n}} \sup_{t_n \le t \le t_{n+1}} \left|\int_0^t [\sigma(s, \xi(s)) - \bar{\sigma}]\, dw(s)\right| > \frac{1}{n}\right\} \\ &\qquad \le \frac{Cn^2}{t_n}(t_{n+1})^{1-\delta} \le \frac{C}{n^2}.\end{aligned}$$

Since, finally,

$$P\left\{\sup_{t_n \le t \le t_{n+1}} |J_3(t)| > \frac{1}{n}\right\} \le \frac{Cn^2}{t_n} E|\xi_0|^2 \le \frac{C}{n^2},$$

we conclude that

$$P\left\{\sup_{t_n \le t \le t_{n+1}} |J(t)| > \frac{1}{n}\right\} < \frac{1}{n^2}.$$

Hence, by the Borel-Cantelli lemma,

$$P\left\{\sup_{t_n \le t \le t_{n+1}} |J(t)| > \frac{1}{n} \text{ i.o.}\right\} = 0;$$

consequently

$$P\left\{\lim_{t\to\infty} |J(t)| = 0\right\} = 1. \tag{4.8}$$

Recalling the law of the iterated logarithm for Brownian motion (see [1], [2]) we obtain from (4.7) the following law:

Theorem 7. *Let the conditions* (2.2), (2.3), (4.5), (4.6) *hold for some* $\delta > 0$, *let* $\bar{\sigma}$ *be nonsingular, and let* $m \ge 2$. *Then, for any* i, $1 \le i \le m$,

$$P\left\{\overline{\lim_{t\to\infty}} \frac{\sum_{j=1}^{m} \hat{\sigma}_{ij}\xi_j(t)}{\sqrt{2t \log \log t}} = 1\right\} = 1, \tag{4.9}$$

$$P\left\{\underline{\lim}_{t\to\infty} \frac{\sum_{j=1}^{m} \hat{\sigma}_{ij}\xi_j(t)}{\sqrt{2t \log \log t}} = -1\right\} = 1. \tag{4.10}$$

Similarly one can prove that

$$P\left\{\overline{\lim_{t\to\infty}} \frac{|\hat{\sigma}\xi(t)|}{\sqrt{2t \log \log t}} = 1\right\} = 1.$$

It was proved by Dvoretsky and Erdös [3] (see also [11]) that if $m \ge 3$ then

$$P\{\lim (|w(t)|/t^{\alpha/2}) = \infty\} = 1 \quad \text{for any } 0 < \alpha < 1.$$

Clearly this holds also for $\bar{\sigma}w(t)$ if $d \ge 3$. But under the assumptions of Theorem 7 (with $\alpha > 1 - \delta$) we get (cf. the proof of (4.8))

$$P\left\{\lim_{t\to\infty} \frac{|\xi(t) - \bar{\sigma}w(t)|}{t^{\alpha/2}} = 0\right\} = 1.$$

Hence,

Theorem 8. *Let the assumptions* (2.2), (2.3), (4.5), (4.6) *hold for some* $\delta > 0$ *and let* $d \ge 3$. *Then*

$$P\left\{\lim_{t\to\infty} \frac{|\xi(t)|}{t^{\alpha/2}} = \infty\right\} = 1 \quad \textit{for any } 0 < \alpha < 1. \tag{4.11}$$

Our last application is to the Cauchy problem for parabolic equations with time independent coefficients. Under some standard conditions (see [4]) there exists a unique solution of the Cauchy problem

$$(4.12)\qquad Lu = \sum_{i,j=1}^{m} b_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{m} a_i(x)\frac{\partial u}{\partial x_i} - \frac{\partial u}{\partial t} = 0 \quad \text{for } 0 < t < \infty,\ x \in R^m,$$

$$(4.13)\qquad u(0, x) = f(x) \quad \text{for } x \in R^m,$$

and it can be written in the form (see [5], [6])

$$(4.14)\qquad u(t, x) = Ef(\xi_x(t))$$

where $\xi_x(t)$ is the solution of (1.1) with $\xi(0) = x$.

These standard conditions are satisfied if

(i) $a_i(x)$, $\sigma_{ij}(x)$ are bounded functions in R^m, locally Lipschitz continuous, uniformly Hölder continuous, and $(b_{ij}(x))$ is uniformly positive definite.

(ii) $f(x) = O(|x|)$ as $|x| \to \infty$ and f is locally Hölder continuous.

The uniqueness of the solution is within the class of functions $z(t, x)$ satisfying, for each $T > 0$,

$$|z(t, x)| \le C \exp\{\beta |x|^2\} \qquad (0 \le t \le T,\ x \in R^m)$$

where C, β are positive constants, depending on T.

Let us further assume that

$$(4.15)\qquad \sum_{i,j} |\sigma_{ij}(x) - \delta_{ij}| = O(|x|^{-\delta}) \quad \text{as } |x| \to \infty,$$

$$(4.16)\qquad \sum_{i} |a_i(x)| = O(|x|^{-1-\delta}) \quad \text{as } |x| \to \infty,$$

and replaced (ii) by the stronger condition

$$(4.17)\qquad |f(x) - f(y)| \le A|x - y| \quad (A \text{ constant}).$$

If $b_{ij} = \frac{1}{2}\delta_{ij}$, $a_i = 0$, then the solution $v(t, x)$ of (4.12), (4.13) is given by $Ef(w(t) + x)$, i.e.

$$(4.18)\qquad v(t, x) = \frac{1}{(2\pi)^{m/2} t^{m/2}} \int_{R^m} \exp\left\{\frac{|x - \xi|^2}{2t}\right\} f(\xi)\, d\xi.$$

Hence, by (4.14), (4.17),

$$|u(t, x) - v(t, x)| \le C\{E|\xi_x(t) - w(t)|^2\}^{1/2} + C.$$

We can now apply Theorems 5(a), 5(c) to estimate the right-hand side. A careful review of the proof of these theorems shows how the estimates depend on the parameter x. This leads to the following theorem.

Theorem 9. *Let the conditions* (i), (4.15), (4.16), (4.17) *hold. If* $m \ge 2$, $0 < \delta < 1$, *then, for all* $t \ge 0$, $x \in R^m$,

$$(4.19)\qquad |u(t, x) - v(t, x)| \le C[t^{(1-\delta)/2} + (1 + |x|)^{1-\delta} + 1] \qquad (C \text{ constant}).$$

If $m \geq 3$ *and* $\delta > 1$ *then, for all* $t \geq 0$, $x \in R^m$,

$$|u(t, x) - v(t, x)| \leq C \quad (C \text{ constant}). \tag{4.20}$$

5. **The case** $m = 1$. In proving Theorem 5(a), or even in deriving (4.3), we have assumed that the matrix $\bar{b}$ has at least two positive eigenvalues. This assumption is essential. To see what may happen when $\bar{b}$ has only one positive eigenvalue, we resort to the case $m = 1$ and, for simplicity, take an equation of the form

$$d\xi(t) = a(\xi(t))\,dt + dw(t). \tag{5.1}$$

We assume that $a(x)$ is continuous and

$$\int_{-\infty}^{\infty} |a(x)|\,dx < \infty. \tag{5.2}$$

One can construct in the present case comparison functions by directly solving equations of the form

$$Lf = \tfrac{1}{2} f''(x) + a(x) f'(x) = \psi(x). \tag{5.3}$$

Setting $A(x) = 2 \int_{-\infty}^{x} a(y)\,dy$, the general solution of (5.3) is given by

$$f(x) = \int_0^x e^{-A(z)}\,dz \left[2 \int_0^z e^{A(y)} \psi(y)\,dy + C_1 \right] + C_2$$

where C_1, C_2 are constants. Using this solution in the proof of Theorem 5, we can derive the estimate

$$E|\xi(t) - w(t)|^2 = o(t) \tag{5.4}$$

provided $A(-\infty) = 0$, i.e.

$$\int_{-\infty}^{\infty} a(x)\,dx = 0. \tag{5.5}$$

If further

$$|a(x)| \leq C/(1 + |x|)^{1+\delta} \quad (0 < \delta < 1), \tag{5.6}$$

then

$$E|\xi(t) - w(t)|^2 = O(t^{1-\delta}). \tag{5.7}$$

If (5.5) is not satisfied then (5.4) is false. In fact, as proved by Kulinič [7], if (5.5) is not satisfied (where $a(x)$ is any continuous function satisfying (5.2)) then already the assertion of Theorem 6 is invalid for any solution $\xi(t)$.

There is an intuitive reason why for $m = 1$, $\sigma \equiv 1$ the assertion of Theorem 6 cannot hold unless (5.5) is satisfied. In order for the distribution of $\xi(t)/\sqrt{t}$ to approximate the normal distribution as $t \to \infty$ the particles represented by $\xi(t)$, or $\xi(t, \omega)$, must be able to move without "resistance" from intervals (α, β) near $+\infty$ to intervals $(-\beta, -\alpha)$. Since in performing this move they must cross the interval $(-\alpha, \alpha)$, they are subject to the influence of the drift term $a(x)$.

This drift coefficient will resist the movement if $\int_{-\infty}^{\infty} a(x)\,dx > 0$; thus this inequality cannot take place. Similarly, the reverse inequality cannot take place.

If $m \geq 2$ and $\bar{b}$ is nonsingular then $\xi(t)$ may move from one m-dimensional interval in a neighborhood G of infinity to another without leaving G. If the drift coefficient $a(t, x)$ is "small" in G (i.e., if $|a(t, x)| \leq C(1 + |x|)^{-1-\mu}$, $\mu > 0$) then there will be negligible resistance by the drift coefficient to the movement of $\xi(t)$ in G. Thus, no condition analogous to (5.5) is required in the case $m \geq 2$.

So far we have considered only the case $\sigma = 1$. The case of general $\sigma(x) > 0$ can be reduced to the previous case by a simple transformation. Setting $\eta(t) = f(\xi(t))$, $f(z) = \int_0^z dy/\sigma(y)$, the equation

$$d\xi(t) = a(\xi(t))\,dt + \sigma(\xi(t))\,dw(t) \tag{5.8}$$

reduces to

$$d\eta = \left[a(g(\eta(t)))/\sigma(g(\eta(t))) - \tfrac{1}{2}\sigma'(g(\eta(t)))\right] dt + dw(t) \tag{5.9}$$

where g is the inverse function of f. The condition (5.5) for the equation (5.9) becomes

$$\int_{-\infty}^{\infty} \frac{a(z) - \frac{1}{2}\sigma(z)\sigma'(z)}{\sigma^2(z)}\,dz = 0. \tag{5.10}$$

If

$$|a(g(y))/\sigma(g(y)) - \tfrac{1}{2}\sigma'(g(y))| \leq C/(1 + |y|)^{1+\delta} \qquad (\delta > 0) \tag{5.11}$$

and if (5.10) holds, then (by the assertion (5.7) applied to the equation (5.9))

$$E|\eta(t) - w(t)|^2 \leq Ct^{1-\delta} + C.$$

Assuming also that

$$|\bar{\sigma}f(x) - x| \leq C(|x|^{1-\mu} + 1) \qquad (0 \leq \mu < 1,\ \bar{\sigma} > 0) \tag{5.12}$$

for some constant $\bar{\sigma}$, we easily derive

$$E|\xi(t) - \bar{\sigma}w(t)|^2 \leq Kt^{1-\nu} + K' \quad \text{for all } t \geq 0, \tag{5.13}$$

where $\nu = \min(\delta, \mu)$ and K, K' are positive constants.

Noting that if

$$|\sigma(x) - \bar{\sigma}| \leq C/(1 + |x|)^{\mu} \tag{5.14}$$

then (5.12) holds, and that (5.11) is equivalent to

$$|a(x)/\sigma(x) - \tfrac{1}{2}\sigma'(x)| \leq C/(1 + |x|)^{1+\delta} \tag{5.15}$$

(provided (5.14) holds), we can state

If (5.15), (5.14) *and* (5.10) *hold, then for any solution* $\xi(t)$ *of* (5.8) $(m = 1)$ *the estimate* (5.13) *holds.*

If (5.14) is replaced by

$$\bar{\sigma} = \lim_{|x| \to \infty} \sigma(x) \text{ exists}, \quad \bar{\sigma} > 0,$$

if the left-hand side of (5.11) is assumed to be absolutely integrable over $(-\infty, \infty)$, and if (5.10) holds, then one can show that

$$E|\xi(t) - \bar{\sigma} w(t)|^2 = o(t). \tag{5.16}$$

Except for Theorem 7, the applications given in §4 extend to the case $m = 1$.

6. **Counterexample.** We shall give an example of a system of m equations $(m \geq 1)$

$$d\xi(t) = a(\xi(t))\,dt + dw(t) \tag{6.1}$$

for which

$$a(x) = O\left(\frac{\log |x|}{|x|}\right) \quad \text{if } |x| \to \infty, \tag{6.2}$$

such that the estimate

$$E|\xi(t)|^2 \leq Kt + K' \quad \text{for all } t \geq 0 \ (K, K' \text{ positive constants}) \tag{6.3}$$

is false.

Let $f(x)$ be a $C^2(R^m)$ function such that

$$f(x) = r^2/\log r \quad \text{if } r = |x| > 2. \tag{6.4}$$

If $a_i(x) = x_i \alpha(r)$ for $|x| > 2$, then

$$\frac{1}{2}\Delta f(x) + \sum_{i=1}^{m} a_i(x) \frac{\partial f(x)}{\partial x_i}$$

$$= \frac{1}{\log r}\left[1 - \frac{m+2}{2}\frac{1}{\log r} + \frac{1}{(\log r)^2}\right] + \frac{2r^2}{\log r}\left[1 - \frac{1}{2 \log r}\right]\alpha(r).$$

Let

$$\alpha(r) = \frac{\log r}{2r^2}[1 + \beta(r)]$$

and define $\beta(r)$ so that

$$\frac{1}{2}\Delta f(x) + \sum_{i=1}^{m} a_i(x) \frac{\partial f(x)}{\partial x_i} = 1 \quad \text{if } |x| > 2. \tag{6.5}$$

It is easily seen that $\beta(r) = O(1)$. We now define $a_i(x)$ in R^m as a continuous function such that

$$a_i(x) = x_i[\log |x| + \beta(|x|)]/2|x|^2 \quad \text{if } |x| > 2. \tag{6.6}$$

By Ito's formula and (6.5),

$$Ef(\xi(t)) = Ef(\xi_0) + E\int_0^1 c(\xi(s))\,ds \tag{6.7}$$

where $c(x) - 1 = 0$ if $|x| > 2$.

We wish to apply the proof of Lemma 1 to $\phi(t, x) = c(x) - 1$. Here the $a_i(t, x) = a_i(x)$ are not bounded by $\epsilon(|x|)/(1 + |x|)$ where $\epsilon(r) \to 0$ if $r \to \infty$. Nevertheless (2.11) takes the form

$$\frac{1}{2}[1 + \theta(r)] \leq \frac{1}{2} m + \sum_i (x_i - e_i) a_i(x) \tag{6.8}$$

where $r = |x - e|$ and $e = (e_1, \cdots, e_m)$ is a point not contained in the support of $c(x) - 1$, i.e. $|e| > 2$. In view of (6.6) and the fact that $\beta(r) = O(1)$, we can construct $\theta(r)$ satisfying (6.8) such that, for any $A > 0$, $\lim_{r\to\infty} \theta(r) = A$. Hence, by the proof of Lemma 1,

$$E\left| \int_0^t [c(\xi(s)) - 1] \, ds \right| = O(1) \quad \text{as } t \to \infty. \tag{6.9}$$

Combining (6.9) with (6.7) we get

$$|Ef(\xi(t)) - t| \leq C_0 \quad \text{for } t \geq 0 \qquad (C_0 \text{ constant}). \tag{6.10}$$

For any $\epsilon > 0$ there is a $B > 0$ such that

$$|x|^2/\log|x| \leq \epsilon |x|^2 + B \quad \text{if } |x| > 2.$$

Hence

$$f(x) \leq \epsilon |x|^2 + C \quad \text{for all } x \in R^m$$

where C is a constant depending on ϵ. This implies

$$Ef(\xi(t)) \leq \epsilon E|\xi(t)|^2 + C. \tag{6.11}$$

Now, if (6.3) holds then from (6.10), (6.11) we obtain

$$t \leq \epsilon K t + \epsilon K' + C + C_0 \quad \text{for all } t \geq 0.$$

But this is impossible if $\epsilon < 1/K$.

If the function $\log|x|$ occurring in (6.2) is replaced by other functions which increase to infinity as $|x| \to \infty$, such that $\log\log|x|$, then we can again show by the above method that (6.3) cannot hold. If, however, $a(x) = O\{1/|x|\}$, then (6.3) holds (by Theorem 2). Recall that if (3.2) holds then (3.1) is also valid.

7. **Equations with unbounded coefficients.** We shall extend some of the results of §§2, 3 to equations with unbounded coefficients.

Theorem 10. *Assume that*

$$\sum_i b_{ii}(t, x) \leq C(1 + |x|^2)^\mu, \qquad \left|\sum x_i a_i(t, x)\right| \leq C(1 + |x|^2)^\mu$$

for some constants $C > 0$, $0 \leq \mu < 1$. Then

$$E|\xi(t)|^{2-2\mu} \leq Kt + K' \quad \textit{for all } t \geq 0, \tag{7.1}$$

where K, K' are positive constants.

Proof. Set $\phi(x) = (1 + |x|^2)^{1-\mu}$. Then $L\phi \le C$ where C is a constant. Using Ito's formula we find that

$$E(1 + |\xi(t)|^2)^{1-\mu} \le Ct + C;$$

this yields (7.1).

To get a lower bound on $E|\xi(t)|^{2-2\mu}$, we set

$$a_i(t, x) = \tilde{a}_i(t, x)(1 + |x|^2)^{\mu}, \tag{7.2}$$

$$\sigma_{ij}(t, x) = \tilde{\sigma}_{ij}(t, x)(1 + |x|^2)^{\mu/2}. \tag{7.3}$$

Then $\tilde{b}(t, x) = \frac{1}{2}\tilde{\sigma}(t, x)\tilde{\sigma}^*(t, x)$ satisfies

$$b_{ij}(t, x) = \tilde{b}_{ij}(t, x)(1 + |x|^2)^{\mu}. \tag{7.4}$$

Lemma 6. *Let the assumptions made in Lemma 1 be satisfied for a_i, b_{ij} replaced by $\tilde{a}_i$, $\tilde{b}_{ij}$. Then for any bounded measurable function $\phi(x)$ with compact support*

$$E\left|\int_0^t \phi(\xi(s))\, ds\right| \le K_1 + K_2 t^{(1+\eta)/(2-2\mu)} \qquad (t \ge 0) \tag{7.5}$$

where K_1, K_2 are constants and η is as in Lemma 1.

Proof. The proof is similar to that of Lemma 1. The only difference is that we introduce ψ such that $|\phi(x)| \le (1 + |x|^2)^{\mu}\psi(|x|)$, and then construct $f(x) = F(r)$ satisfying $Lf(x) \ge (1 + |x|^2)^{\mu}\psi(|x|)$. We obtain

$$E\left|\int_0^t \phi(\xi(s))\, ds\right| \le CE|\xi(t)|^{1+\eta},$$

and then use Theorem 10.

If $\phi(x)$ does not have a compact support, but

$$|\phi(x)| \le C(1 + |x|^2)^{\mu}/(1 + |x|)^{\alpha},$$

then we can proceed as in Lemma 2. We find that

$$E\left|\int_0^t \phi(\xi(s))\, ds\right| \le Ct^{(1+\eta)/(2-2\mu)} + Ct^{(2-\alpha)/(2-2\mu)}.$$

Theorem 11. *Let (2.2)–(2.5) hold with a_i, b_{ij} replaced by $\tilde{a}_i$, $\tilde{b}_{ij}$, and let $0 \le \mu < 1$. Then*

$$E|\xi(t)|^{2-2\mu} \ge Kt - K' \quad \text{for all } t \ge 0, \tag{7.6}$$

where K, K' are positive constants.

Proof. Note that

$$L(1 + |x|^2)^{\mu} \ge \gamma - \phi(x)$$

where $\gamma > 0$ and $\phi(x)$ has compact support, and apply Lemma 6.

We finally note that using the result preceding Theorem 11 one can proceed to establish an analog of Theorem 5. A typical estimate has the form

$$E\left|\xi(t) - \bar{\sigma}\int_0^t [1 + |\xi(s)|^2]^{\mu}\, dw(s)\right|^2 = O(t^{2/(2-2\mu)-\epsilon}) \tag{7.7}$$

for some $\epsilon > 0$, where $\bar{\sigma} = \lim_{|x|\to\infty} \tilde{\sigma}(t, x)$. The estimate (7.7) does not appear to be interesting when $\mu > 0$.

The case $\mu < 0$ can be treated in a similar fashion. The assertion (7.1) remains true for $\mu < 0$ provided

$$\sum_{i,j} |b_{ij}(t, x)| + \left|\sum_i x_i a_i(t, x)\right| \le C(1 + |x|^2)^{\mu},$$

and the assertion (7.6) remains true under the same assumptions as in Theorem 11.

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